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HIGH-ORDER BEM FORMULATIONS FOR STRONGLY NON-LINEAR PROBLEMS GOVERNED BY QUITE GENERAL NON-LINEAR DIFFERENTIAL OPERATORS. PART 2: SOME 2D EXAMPLES

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SUMMARY

In this paper the general BEM proposed previously by Liao is applied to solve some 2D strongly non-linear differential equations, even including those whose governing equations and boundary conditions do not contain any linear terms. It is shown that the proposed general BEM is really valid for general non-linear problems, so that it can be applied to solve high-dimensional, strongly non-linear problems in engineering. © 1997 by John Wiley & Sons, Ltd. Int. j. numer. methods fluids 24: 863–873, 1997.

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KEY WORDS: general BEM; general non-linear differential operator; homotopy

1. INTRODUCTION

The boundary element method[1–4] (BEM) is in principle based on the linear superposition of the fundamental solution of a linear operator. Nowadays, many researchers[5–6] apply the BEM to solve non-linear problems. The basic idea of the current BEM for nonlinear problems is to move all non-linear terms to the right-hand side of the equations and then find the corresponding fundamental solutions of the linear operator remaining on the left-hand side of the equations. In the BEM for non-linear problems, iteration is necessary and a domain integral term appears.

The above-mentioned BEM for non-linear differential equations has some obvious restrictions. First of all, it is invalid if nothing is left after moving all non-linear terms of an equation to its right-hand side, i.e. the equation does not contain any linear terms so that there certainly does not exist a fundamental solution at all. Secondly, even if a linear operator exists, it may be so simple that it can not satisfy all boundary conditions. Finally, this linear operator might be so complex that its fundamental solution is unknown or quite difficult to find. In the first two cases the traditional BEM for non-linear problems does not work at all. In the last case it is not easy to apply the BEM.

Liao[7,8] proposed a new kind of BEM for quite general non-linear differential equations even including those whose governing equations and boundary conditions do not contain any linear terms at all. This general BEM can overcome the three above-mentioned restrictions of the traditional BEM for non-linear problems. It is based on homotopy in topology so that it has a solid mathematical base.

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This general BEM offers great freedom in selecting the linear operator and initial approximation. Some examples of *one-dimensional* highly non-linear differential equations are given in References 7 and 8. The high-order BEM formulaes for general governing equations and boundary conditions are given in Reference 8.

This paper is the continuation of the author's work described in Reference 8. In this paper the general BEM proposed by Liao in References 7 and 8 is further applied to solve some 2D strongly non-linear problems whose governing equation and boundary conditions do not contain any linear terms at all. The purpose of this paper is to show that the proposed general BEM is really valid for *high-dimensional* strongly non-linear differential equations.

2. BASIC IDEAS OF THE PROPOSED GENERAL BEM

Consider the non-linear differential equation

$$A(u) = f(\vec{r}), \quad \vec{r} \in \Omega, \tag{1}$$

with boundary conditions

$$H\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad \vec{r} \in \Gamma, \tag{2}$$

where A is a general differential operator, $f(\vec{r})$ is a known function of the co-ordinates of the point $\vec{r} \in \Omega$ and H is a function of u and its derivatives $\partial u/\partial n$ on the boundary Γ of the domain Ω. For simplicity, we define $u' = \partial u/\partial n$ (on the boundary Γ) in this paper.

In References 7 and 8, by means of constructing a homotopy[9] $v(\vec{r}, p) : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies

$$(1-p)[L(v) - L(u_0)] + p[A(v) - f(\vec{r})] = 0, \quad p \in [0, 1], \quad \vec{r} \in \Omega, \tag{3}$$

with boundary condition

$$H(v, v') = (1-p)H(u_0, u_0'), \quad p \in [0, 1], \quad \vec{r} \in \Gamma, \tag{4}$$

we obtain a *family* of iterative BEM formulae at high order,

$$u_{k+1}(\vec{r}) = u_k(\vec{r}) + \sum_{m=1}^{M} \left(\frac{v_0^{[m]}(\vec{r})}{m!} \right) \lambda^m \quad (k = 0, 1, 2, 3, \ldots), \tag{5}$$

where L is a properly selected *linear* operator whose fundamental solution is known, $u_0(\vec{r})$ is an initial approximation which can be selected with great freedom, $p \in [0, 1]$ is an imbedding parameter and $v(\vec{r}, p)$ is now a function of both $\vec{r} \in \Omega$ and $p \in [0, 1]$. The term

$$v_0^{[m]}(\vec{r}) = \left. \frac{\partial^m v(\vec{r}, p)}{\partial p^m} \right|_{p=0}$$

is called the *mth-order deformation derivative* and is determined by the so-called *mth-order deformation equation*

$$L(v_0^{[m]}) = f_m(\vec{r}), \quad \vec{r} \in \Omega \quad (m = 1, 2, 3, \cdots), \tag{6}$$

with boundary condition

$$\left. \frac{\partial H}{\partial u} \right|_{p=0} v_0^{[m]}(\vec{r}) + \left. \frac{\partial H}{\partial u'} \right|_{p=0} \frac{\partial v_0^{[m]}(\vec{r})}{\partial n} = h_m(\vec{r}), \quad \vec{r} \in \Gamma \quad (m = 1, 2, 3, \cdots), \tag{7}$$

where

$$f_1(\vec{r}) = f(\vec{r}) - A(u_0), \tag{8}$$

$$f_m(\vec{r}) = m\left(L(v_0^{[m-1]}) - \left.\frac{\mathrm{d}^{m-1}[A(v)]}{\mathrm{d}p^{m-1}}\right|_{p=0}\right) \quad (m > 1) \tag{9}$$

and

$$h_1(\vec{r}) = -H(u_0, u_0'), \tag{10}$$

$$h_2(\vec{r}) = -\left.\left[\frac{\partial^2 H}{\partial u^2}(v_0^{[1]})^2 + 2\frac{\partial^2 H}{\partial u \partial u'} v_0^{[1]}\frac{\partial v_0^{[1]}}{\partial n} + \frac{\partial^2 H}{\partial (u')^2}\left(\frac{\partial v_0^{[1]}}{\partial n}\right)^2\right]\right|_{p=0}, \tag{11}$$

Note that equation (6) is *linear* with *linear* boundary condition (7). Moreover, the linear operator L can be properly selected so that its fundamental solution is known. Therefore the mth-order deformation equation (6) under condition (7) can be easily solved by the traditional BEM in the following way:

$$c(\vec{r})v_0^{[m]}(\vec{r}) = \int_\Gamma [v_0^{[m]}B(\omega) - \omega B(v_0^{[m]})]\mathrm{d}\Gamma + \int_\Omega f_m \omega \mathrm{d}\Omega, \tag{12}$$

where B is the corresponding boundary operator for the freely selected linear operator L, ω is the fundamental solution of L and $c(\vec{r})$ is a known coefficient dependent upon the co-ordinates of the point $\vec{r}$. The detailed formulations of the above equations are given in Reference 7 and 8.

After selecting an initial approximation $u_0(\vec{r})$, the term $f_m(\vec{r})$ on the right-hand side of equation (6) is known for each m ($m \geqslant 1$). Note that we now have very great freedom to select the corresponding linear operator L, or more precisely, we can now select a proper linear operator L whose fundamental solution is known even if the non-linear operator A under consideration does *not* contain any linear terms at all. In the special case where the operator A can be divided into two parts, one linear, the other non-linear, so that $A = \hat{L} + \hat{N}$ holds, and moreover, $\hat{L}$ is proper and used as the linear operator whose fundamental solution is $\hat{\omega}$, then the above-mentioned general BEM formula in case $M = 1$ becomes

$$\begin{aligned} c(\vec{r})v_0^{[1]}(\vec{r}) &= \int_\Gamma [v_0^{[1]}\hat{B}(\hat{\omega}) - \hat{\omega}\hat{B}v_0^{[1]}]\mathrm{d}\Gamma + \int_\Omega [f - \hat{L}(u) - \hat{N}(u)]\hat{\omega}\mathrm{d}\Omega \\ &= \int_\Gamma [v_0^{[1]}\hat{B}(\hat{\omega}) - \hat{\omega}\hat{B}(v_0^{[1]})]\mathrm{d}\Gamma + \int_\Omega [f - \hat{N}(u)]\hat{\omega}\mathrm{d}\Omega - c(\vec{r})u(\vec{r}) + \int_\Gamma [u\hat{B}(\hat{\omega}) - \hat{\omega}\hat{B}(u)]\mathrm{d}\Gamma, \end{aligned} \tag{13}$$

which gives

$$c(\vec{r})[u(\vec{r}) + v_0^{[1]}(\vec{r})] = \int_\Gamma [(u + v_0^{[1]})\hat{B}(\hat{\omega}) - \hat{\omega}\hat{B}(u + v_0^{[1]})]\mathrm{d}\Gamma + \int_\Omega [f(\vec{r}) - \hat{N}(u)]\hat{\omega}\mathrm{d}\Omega, \quad \vec{r} \in \Omega. \tag{14}$$

Let $u_k(\vec{r}) = u(\vec{r})$ and $\tilde{u}_k(\vec{r}) = u(\vec{r}) + v_0^{[1]}(\vec{r})$. The above expression can be rewritten as

$$c(\vec{r})\tilde{u}_k(\vec{r}) = \int_\Gamma [\tilde{u}_k\hat{B}(\hat{\omega}) - \hat{\omega}\hat{B}(\tilde{u}_k)]\mathrm{d}\Gamma + \int_\Omega [f(\vec{r}) - \hat{N}(u_k)]\hat{\omega}\mathrm{d}\Omega, \tag{15}$$

which is exactly the formula of the traditional BEM for non-linear problems. Moreover, if a relaxation parameter λ is introduced into the iteration, we have

$$\begin{aligned} u_{k+1}(\vec{r}) &= u_k(\vec{r}) + \lambda[\tilde{u}_k(\vec{r}) - u_k(\vec{r})] \\ &= u_k(\vec{r}) + \lambda v_0^{[1]}(\vec{r}), \end{aligned} \tag{16}$$

which is exactly formula (5) in the case $M = 1$! This means that the traditional boundary element method for non-linear problems is indeed only a special case of the newly proposed boundary element method, so that our above-mentioned BEM is more general.

Finally we emphasize once again that the operator A in (1) and the operator H in (2) are quite general. All the above-mentioned formulae are valid even if both the operator A and the operator H do not contain any linear terms. This means that the newly proposed boundary element method is still valid no matter whether or not there exist linear terms in the original governing equation (1) and boundary condition (2). Therefore it is possible for us to use the newly proposed boundary element method to solve more non-linear problems with strong non-linearity.

3. NUMERICAL EXAMPLES

In References 7 and 8, we showed that the proposed general BEM is valid for quite highly non-linear problems. However, the examples given in References 7 and 8 are only one-dimensional and are generally considered to be not satisfactory. Hence in this paper we apply the proposed general BEM to solve some 2D non-linear problems in order to show that the general BEM is indeed valid for high-dimensional non-linear problems.

Example 1

Consider the 2D second-order non-linear differential equation

$$\tfrac{1}{4}[(u_{xx})^2 + (u_{yy})^2] + \tfrac{1}{2}u_x u_y = \mathrm{e}^{-2(x+y)}, \quad x \in [0, 1], \quad y \in [0, 1], \tag{17}$$

with boundary conditions

$$u(x, y) = \mathrm{e}^{-y}, \quad x = 0, \quad y \in [0, 1], \tag{18}$$

$$\frac{\partial u(x, y)}{\partial n} = \mathrm{e}^{-x}, \quad x \in [0, 1], \quad y = 0, \tag{19}$$

$$u(x, y) + \frac{\partial u(x, y)}{\partial n} = 0, \quad x = 1, \quad y \in [0, 1], \tag{20}$$

$$u(x, y)\cos\left(\frac{\partial u(x, y)}{\partial n}\right) + \frac{\partial u(x, y)}{\partial n}\cos[u(x, y)] = 0, \quad x \in [0, 1], \quad y = 1. \tag{21}$$

It should be emphasized that the governing equation (17) does not contain any linear terms at all so that the traditional BEM is invalid. Note that (18) is a Dirichlet-type condition, (19) is a Neumann-type condition, (20) is a mixed-type condition and (21) is a condition which does not contain any linear terms. Note also that $u(x, y) = \exp(-x - y)$ is one of the solutions of the above problem so that we can compare our numerical approximation with it.

For all examples considered in this paper, we use the 2D Laplace operator

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{22}$$

as the linear operator. For simplicity we always use $u_0(x, y) = 0$ as the initial approximation. For the numerical domain integral we divide the domain $\Omega = [0, 1] \times [0, 1]$ into $N_\Omega \times N_\Omega$ equal subdomains and each boundary into N_Γ equal elements in which the unknowns are linearly distributed. At each corner, two very close points on different boundaries are used to treat the discontinuity of the unknowns at corners. Throughout this paper we use $N_\Omega = N_\Gamma = 40$. In order to check whether iteration procedures converge or not, we investigate the following two kinds of root-mean-square errors:

$$RMS_1 = \sqrt{\left(\sum_{i=0}^{N_\Omega}\sum_{j=0}^{N_\Omega}\frac{[u_k(x_i, y_j) - \mathrm{e}^{-(x_i+y_j)}]^2}{(1+N_\Omega)^2}\right)} \quad \left(x_i = \frac{i}{N_\Omega}, y_j = \frac{j}{N_\Omega}\right), \tag{23}$$

$$RMS_2 = \sqrt{\left(\sum_{i=0}^{N_\Omega}\sum_{j=0}^{N_\Omega}\frac{[u_{k+1}(x_i, y_j) - u_k(x_i, y_j)]^2}{(1+N_\Omega)^2}\right)} \quad \left(x_i = \frac{i}{N_\Omega}, y_j = \frac{j}{N_\Omega}\right), \tag{24}$$

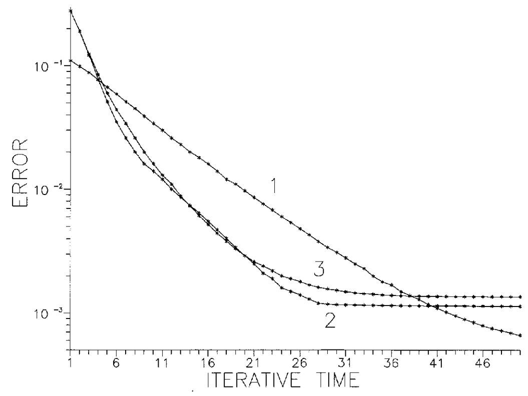

Figure 1. Errors of solution (RMS_1 defined by (23)) versus iterative time for Examples 1–3

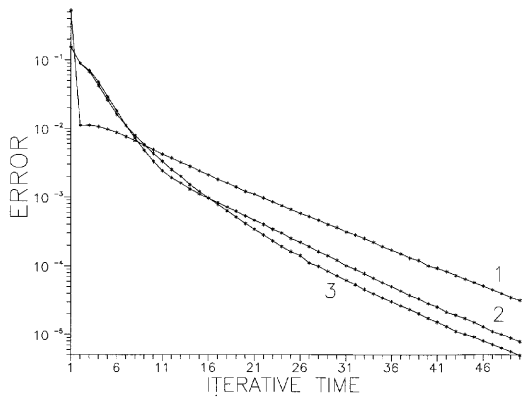

Figure 2. Errors of solution (RMS_2 defined by (24)) versus iterative time for Examples 1–3

We simply apply the first-order ($M = 1$) expression (5) as our iterative formula and use $\lambda = 1$ as the iterative parameter. The iteration converges quickly to the exact solution $\exp(-x-y)$, as shown in Figures 1 and 2. We should emphasize that the governing equation (17) does *not* contain any linear terms so that the traditional BEM is invalid for it. However, the proposed general BEM works quite well. This example illustrates that the general BEM is indeed valid for high-dimensional non-linear problems whose governing equation does not contain any linear terms.

Example 2

In the second example we consider again the same governing equation (17) but with more complex non-linear boundary conditions

$$u\cos\left(\frac{\partial u}{\partial n}\right) - \frac{\partial u}{\partial n}\cos(u) + \sin^2(u) + \cos^2\left(\frac{\partial u}{\partial n}\right) = 1, \quad x = 0, \quad y \in [0,1], \tag{25}$$

$$u\cos\left(\frac{\partial u}{\partial n}\right) - \frac{\partial u}{\partial n}\cos(u) + \sin^2(u) + \cos^2\left(\frac{\partial u}{\partial n}\right) = 1, \quad x \in [0,1], \quad y = 0, \tag{26}$$

$$u\cos\left(\frac{\partial u}{\partial n}\right) + \frac{\partial u}{\partial n}\cos(u) + u\frac{\partial u}{\partial n} + e^{-2(1+x)} = 0, \quad x \in [0,1], \quad y = 1, \tag{27}$$

$$u\cos\left(\frac{\partial u}{\partial n}\right) + \frac{\partial u}{\partial n}\cos(u) + u\frac{\partial u}{\partial n} + e^{-2(1+y)} = 0, \quad x = 1, \quad y \in [0,1]. \tag{28}$$

We should emphasize that both the governing equation (17) and all four boundary conditions (25)–(28) do *not* contain any linear terms at all! Note also that $u(x,y) = \exp(-x-y)$ is one of the solutions of the above problem too.

We simply use the first-order ($M = 1$) expression (5) as our iterative formula and use $\lambda = 1$ as the iterative parameter. Once again we obtain a numerical result which converges to the exact solution $\exp(-x-y)$. The corresponding errors versus iterative times are shown in Figures 1 and 2. This example indicates that the proposed general BEM is valid even for those non-linear problems whose governing equations and boundary conditions do *not* contain any linear terms at all!

Example 3

Finally, we consider the more complex non-linear differential equation

$$\tfrac{1}{4}[(u_{xx})^2 + (u_{yy})^2] + \tfrac{1}{2}u_x u_y + \ln\left(\frac{1 + \sin^2(u_{xx}u_{yy}) + \cos^2(u_x u_y)}{2}\right) = \alpha e^{-2(x+y)},$$
$$x \in [0,1], \quad y \in [0,1], \alpha \in (0,\infty), \tag{29}$$

with the same boundary conditions (25)–(28). Note that both the governing equation and boundary conditions do not contain any linear terms. In particular the governing equation (29) contains trigonometric and logarithmic functions of the non-linear expressions of the unknown $u(x,y)$, so that the non-linearity of the governing equation (29) is quite high.

In the case $\alpha = 1$, the real function $\exp(-x-y)$ is one of the solutions of the third example. In the case $\alpha = 1$ we simply apply the first-order ($M = 1$) expression (5) as our iterative formula and use $\lambda = 1$ as the iterative parameter. Once again, we successfully obtain a numerical result which converges to the exact solution $\exp(-x-y)$, as shown in Figures 1 and 2. The numerical results of u and $\partial u/\partial n$ on the four boundaries agree quite well with the corresponding values of the exact solution, as shown in Figures 3 and 4. Note that we use $u_0(x,y) = 0$ as the initial approximation. This example

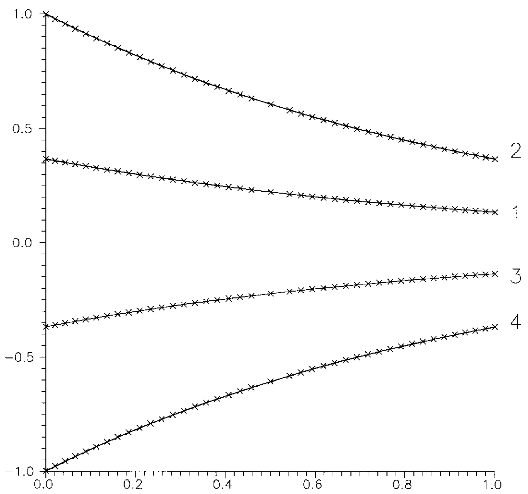

Figure 3. Comparison of numerical results on boundaries with corresponding exact solution of Example 3 in case $\alpha = 1$: curve 1, u on boundary $x \in [0, 1], y = 1$; curve 2, u on boundary $x \in [0, 1], y = 0$; curve 3 $\partial u/\partial n$ on boundary $x \in [0, 1], y = 0$; curve 4, $-\partial u/\partial n$ on boundary $x \in [0, 1], y = 0$; crosses, corresponding values of exact solution

indicates that the proposed general BEM is valid even for quite complex, highly non-linear differential equations with quite complex non-linear boundary conditions. Note that both the governing equation and boundary conditions of the third example do not contain any linear terms.

In the case $\alpha \neq 1$ we also obtain convergent numerical results which are different from $\exp(-x - y)$, as shown in Figures 5–8. The numerical parameters for the third example are given in Table I. We apply both the first-order ($M = 1$) and second-order ($M = 2$) iterative formulae of (5). We find that if the second-order iterative formula ($M = 2$) converges under a value of $\lambda = \mu$, then it

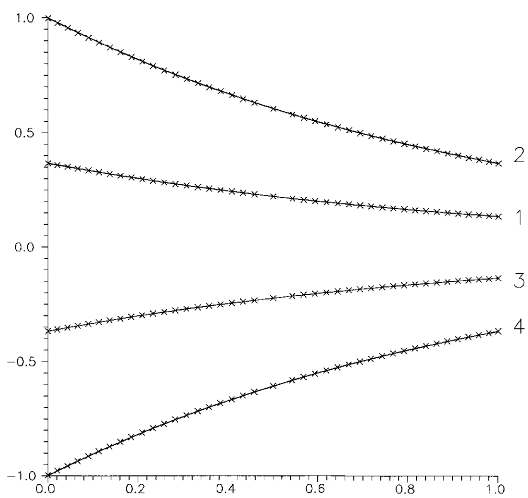

Figure 4. Comparison of numerical results on boundaries with corresponding exact solution of Example 3 in case $\alpha = 1$: curve 1, u on boundary $y \in [0, 1], x = 1$; curve 2, u on boundary $y \in [0, 1], x = 0$; curve 3 $\partial u/\partial n$ on boundary $x \in [0, 1], x = 1$; curve 4, $-\partial u/\partial n$ on boundary $y \in [0, 1], x = 0$; crosses, corresponding values of exact solution

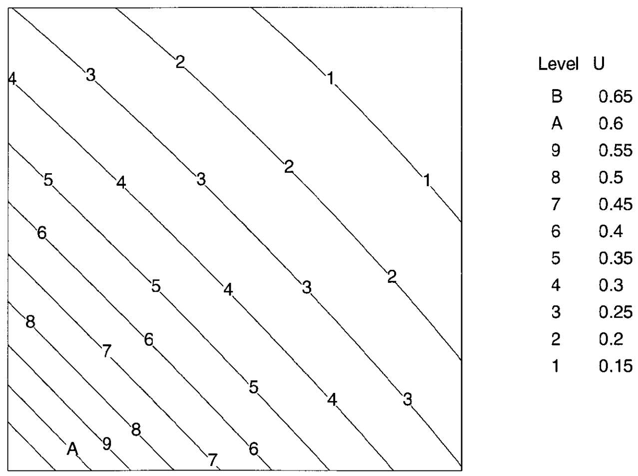

Figure 5. Solution of Example 3 in case $\alpha = 0{\cdot}5$

converges faster than by the first-order formula ($M = 1$) under the same value of $\lambda = \mu$. Moreover, if the iteration converges under a value of $\lambda = \mu_1$, then the same iterative formula under a value of $\lambda = \mu_2 < \mu_1$ converges more slowly. In Reference 8, we show similar results for one-dimensional examples.

It should be emphasized that only *one* linear operator, i.e. the 2D Laplace operator, and its corresponding fundamental solution are used for *all* the above quite different non-linear problems. This is very interesting and deserves further research in detail. It seems that a fairly general BEM computer programme might be developed to solve a large number of quite different sorts of high-dimensional strongly non-linear problems in engineering, especially when the proposed general BEM is combined with the well-established dual reciprocity method that can transform the domain integral into the surface so that much less CPU capacity is necessary.

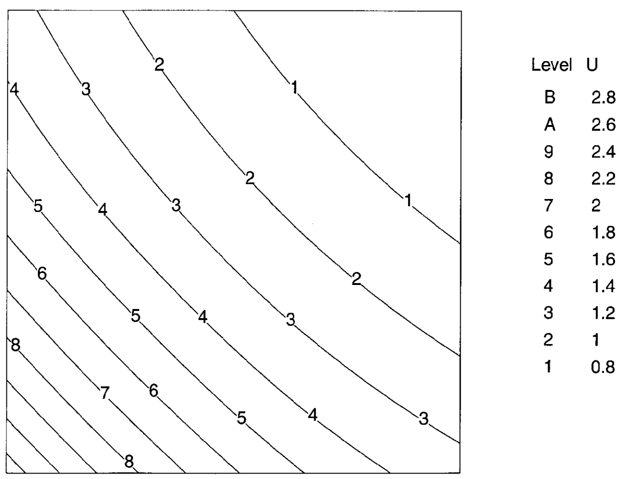

Figure 6. Solution of Example 3 in case $\alpha = 10$

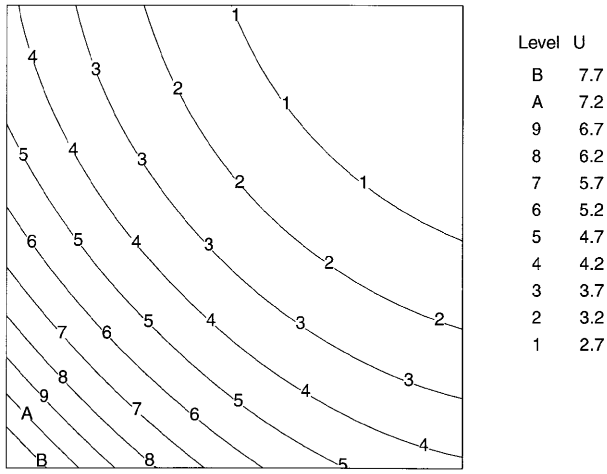

Figure 7. Solution of Example 3 in case $\alpha = 100$

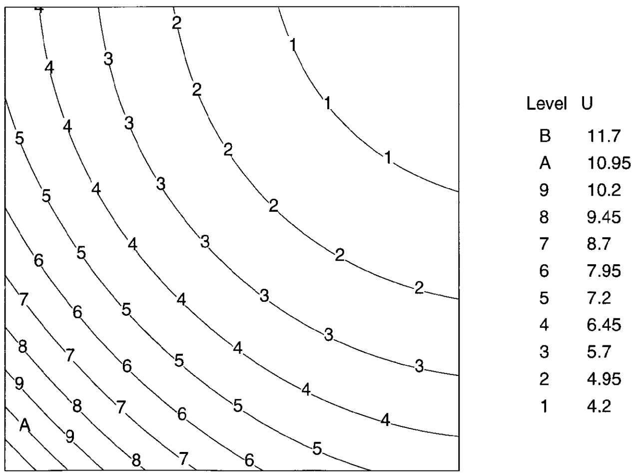

Figure 8. Solution of Example 3 in case $\alpha = 250$

Table I. Numerical parameters used for Example 3

α	λ
0	0·10
0·5	0·25
5	0·50
10	0·50
50	0·25
100	0·25
250	0·10

4. CONCLUSIONS AND DISCUSSION

In this paper the general boundary element method for strongly non-linear problems proposed by Liao[7,8] is proved to be valid for quite complex, strongly non-linear 2D differential equations, even including those whose governing equations and boundary conditions do *not* contain any linear terms at all. Based on these examples, we have many reasons to believe that the proposed general BEM can be applied to solve high-dimensional strongly non-linear problems in engineering. Note that the proposed general BEM has been successfully applied to solve viscous flows governed by the N–S equations,[5,10] the non-linear heat transfer of inhomogeneous materials, etc.

It should be emphasized that we use the 2D Laplace operator as the linear operator for all three quite different non-linear problems under consideration. This is very interesting. It implies that a general BEM software for different types of strongly non-linear problems might be developed, because a simple linear operator whose fundamental solution is known might be used for a large number of quite different types of non-linear differential equations, as illustrated in this paper.

From the theoretical viewpoint it seems that the proposed general BEM might overcome nearly all restrictions of the traditional BEM for non-linear problems and could be applied to solve reasonable non-linear differential equations. However, as in the traditional BEM for non-linear problems, the domain integral term appears in the general BEM, which decreases greatly the effectiveness of the proposed general BEM. There might exist two ways to overcome this disadvantage of the proposed general BEM. One is to use a vector supercomputer, because the parallel process is especially simple and quite effective for the integral. The other is to apply the so-called dual reciprocity method[11] which was developed to increase the effectiveness of the traditional BEM for non-linear problems by means of transforming the domain integral onto the surface. Both deserve further researches in detail.

Finally we would like to point out that the general BEM proposed in References 7 and 8 has a solid mathematical base. In fact, it is only a simple application of a newly proposed non-linear analytical technique, the homotopy analysis method,[12,13] which is based on homotopy in topology and has been successfully applied to solve many non-linear problems. The author has even applied the homotopy analysis method to obtain some wonderful results in pure mathematics, such as the generalized Newtonian binomial theorem about $(1+t)^a$ for fractional and negative exponents which has been rigorously proved to be valid even in the region $t \in [-1, \infty)$ and the generalized Taylor formula which can give a family of power series of a real function $f(t)$ whose convergence radius can be much greater than that of the traditional Taylor series of $f(t)$. All these results give us confidence to believe the reasonableness of both the homotopy analysis method[12,13] and the general boundary element method.[7,8]

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